Commutativity and Spectra of Hermitian Matrices

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ABSTRACT

If two Hermitian matrices commute, then the eigenvalues of their sum are just the sums of the eigenvalues of the two matrices in a suitable order. Examples show that the converse is not true in general. In this paper, partial converses are obtained. The technique involves a characterization of the equality cases for Weyl's inequalities. Moreover, a new proof on the commutativity of two Hermitian matrices with property L and analogous results for the product of two positive definite Hermitian matrices are included.

1. EQUALITY CASES OF WEYL'S INEQUALITIES

Weyl's inequalities for the sum of two Hermitian matrices are usually proved by the minimax principle; for example see [2, p. 181]. Another approach is suggested by Ikebe, Inagaki, and Miyamoto in [3]. The advantage of the latter approach is that it makes the equality cases transparent. We will reproduce their proof to illustrate this point. First we state two key ideas as the following lemmas; their proofs are standard.

LEMMA 1.1. *Let S_1, S_2, and S_3 be subspaces in the n-dimensional complex Euclidean space $\mathbf{C}^n$. Then* $\dim(S_1 \cap S_2 \cap S_3) \geq \dim S_1 + \dim S_2 + \dim S_3 - 2n$.

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LEMMA 1.2. *Let H be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and corresponding orthonormal eigenvectors $x_1, \ldots, x_n$. For a strictly increasing integer sequence $1 \leq i_1 < \cdots < i_m \leq n$, define $S = span\{x_{i_1}, \ldots, x_{i_m}\}$. If $x \in S - \{0\}$, then*

$$\lambda_{i_m} \leq \frac{x^* H x}{x^* x} \leq \lambda_{i_1}.$$

Moreover, equality in the right-hand (respectively, left-hand) inequality holds if and only if $Hx = \lambda_{i_1} x$ (respectively, $Hx = \lambda_{i_m} x$).

From now on, if an $n \times n$ matrix X has real eigenvalues, then its non-increasingly ordered eigenvalues are denoted by $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$.

THEOREM 1.3. *Let A and B be $n \times n$ Hermitian matrices.*

(i) *For integers i and j such that $1 \leq i, j, i+j-1 \leq n$,*

$$\lambda_{i+j-1}(A+B) \leq \lambda_i(A) + \lambda_j(B).$$

Moreover, equality holds if and only if there exists a unit vector x such that $Ax = \lambda_i(A)x, Bx = \lambda_j(B)x$, and $(A+B)x = \lambda_{i+j-1}(A+B)x$.

(ii) *For integers i and j such that $1 \leq i, j, i+j-n \leq n$,*

$$\lambda_{i+j-n}(A+B) \geq \lambda_i(A) + \lambda_j(B).$$

Moreover, equality holds if and only if there exists a unit vector x such that $Ax = \lambda_i(A)x, Bx = \lambda_j(B)x$, and $(A+B)x = \lambda_{i+j-n}(A+B)x$.

Proof. Let $u_1, \ldots, u_n$ denote orthonormal eigenvectors of A corresponding to the respective eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$; $v_1, \ldots, v_n$ denote orthonormal eigenvectors of B corresponding to the respective eigenvalues $\lambda_1(B), \ldots, \lambda_n(B)$; and $w_1, \ldots, w_n$ denote orthonormal eigenvectors of $A+B$ corresponding to the respective eigenvalues $\lambda_1(A+B), \ldots, \lambda_n(A+B)$.

(i): Consider subspaces $S_1 = \text{span}\{u_i, \ldots, u_n\}$, $S_2 = \text{span}\{v_j, \ldots, v_n\}$, and $S_3 = \text{span}\{w_1, \ldots, w_{i+j-1}\}$. By Lemma 1.1, $\dim(S_1 \cap S_2 \cap S_3) \geq (n+1-i) + (n+1-j) + (i+j-1) - 2n = 1$. Hence there exists a unit vector $x \in S_1 \cap S_2 \cap S_3$. Using Lemma 1.2,

$$\lambda_{i+j-1}(A+B) \leq x^*(A+B)x = x^*Ax + x^*Bx \leq \lambda_i(A) + \lambda_j(B).$$

If $\lambda_{i+j-1}(A+B) = \lambda_i(A) + \lambda_j(B)$ then $x^*Ax = \lambda_i(A), x^*Bx = \lambda_j(B)$, and $x^*(A+B)x = \lambda_{i+j-1}(A+B)$. Hence, by Lemma 1.2 again, x

is a common eigenvector for A, B, and $A+B$ with respect to the appropriate eigenvalues. The converse is obvious.

(ii): Consider subspaces $S_1 = \text{span}\{u_1, \ldots, u_i\}, S_2 = \text{span}\{v_1, \ldots, v_j\}$, and $S_3 = \text{span}\{w_{i+j-n}, \ldots, w_n\}$. By Lemma 1.1, $\dim(S_1 \cap S_2 \cap S_3) \geq i + j + (2n + 1 - i - j) - 2n = 1$. Hence there exists a unit vector $x \in S_1 \cap S_2 \cap S_3$. The rest of the proof proceeds in a similar line of argument to (i).

■

If X is a Hermitian matrix, then $\lambda_k(-X) = -\lambda_{n+1-k}(X)$ for all $k = 1, \ldots, n$. With this observation, (ii) of Theorem 1.3 also follows from applying (i) to the matrices $-A$ and $-B$.

2. COMMUTATIVITY AND SPECTRUM

Throughout this section, A and B always denote Hermitian matrices. A and B commute if and only if they share a full set of common orthonormal eigenvectors [2, p. 235]. Hence the eigenvalues of their sum are easy to obtain.

THEOREM 2.1. *If $AB = BA$ then there exist permutations a and b of $\{1, \ldots, n\}$ such that $\lambda_k(A + B) = \lambda_{a(k)}(A) + \lambda_{b(k)}(B)$ for all $k = 1, \ldots, n$.*

However, the converse is not true in general. An example is given to illustrate this fact. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6-\sqrt{2} & 2 \\ 0 & 2 & 4+2\sqrt{2} \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the spectra of A, B and $A+B$ are respectively $\{8, 2+\sqrt{2}, 0\}$, $\{4, 0, -4\}$, and $\{6+\sqrt{2}, 4, 0\}$. Hence $6+\sqrt{2} = (2+\sqrt{2}) + 4$, $4 = 8 + (-4)$, $0 = 0 + 0$, but $AB \neq BA$. Nevertheless we do have some partial converses. Their proofs use a lemma that consists of special cases of Theorem 1.3.

LEMMA 2.2.

(i) *Given $1 \leq k \leq n$. If $\lambda_k(A + B) = \lambda_k(A) + \lambda_1(B)$, then there exists a unit vector x such that $Ax = \lambda_k(A)x$ and $Bx = \lambda_1(B)x$.*

(ii) *Given $1 \leq k \leq n-1$. If $\lambda_{k+1}(A + B) = \lambda_k(A) + \lambda_2(B)$, then there exists a unit vector x such that $Ax = \lambda_k(A)x$ and $Bx = \lambda_2(B)x$.*

THEOREM 2.3. *Let rank $A = 1$ or rank $B = 1$. If there exist permutations a and b of $\{1, \dots, n\}$ such that $\lambda_k(A+B) = \lambda_{a(k)}(A) + \lambda_{b(k)}(B)$ for all $k = 1, \dots, n$, then $AB = BA$.*

Proof. Without loss of generality, we can assume $\operatorname{rank} B = 1$ and its only nonzero eigenvalue is positive, i.e., $\lambda_1(B) > 0$ and $\lambda_2(B) = \cdots = \lambda_n(B) = 0$. Now we proceed by induction on n, the dimension of the matrices. For $n = 1$, there is nothing to prove. Assume the statement is true when the dimension of the matrices is $n-1$. The rest of the proof is divided into two mutually exclusive cases.

Case 1: $\lambda_{k_0}(A+B) = \lambda_{k_0}(A) + \lambda_1(B)$ for some $k_0 \geq 1$. Using Lemma 2.2(i), there exists a common unit eigenvector x for A and B associated with the eigenvalues $\lambda_{k_0}(A)$ and $\lambda_1(B)$, respectively. Find a unitary matrix U with x as its first column. Then

$$U^*AU = \begin{bmatrix} \lambda_{k_0}(A) & 0 \\ 0 & C \end{bmatrix}, \qquad U^*BU = \begin{bmatrix} \lambda_1(B) & 0 \\ 0 & D \end{bmatrix},$$

where C and D are $(n-1) \times (n-1)$ Hermitian matrices. Notice that D is the $(n-1) \times (n-1)$ zero matrix, since rank $B = 1$ and $\lambda_1(B) \neq 0$. Hence $CD = DC = 0$, so $AB = BA$.

Case 2: $\lambda_k(A+B) \neq \lambda_k(A) + \lambda_1(B)$ for all $k > 1$. Let $b(i) = 1$, i.e., $\lambda_i(A+B) = \lambda_{a(i)}(A) + \lambda_1(B)$. For $k \neq i$, $\lambda_{b(k)}(B) = 0$ and so $\lambda_k(A+B) = \lambda_{a(k)}(A)$. Hence a can be chosen such that $a(1) < \cdots < \widehat{a(i)} < \cdots < a(n)$, where $\widehat{a(i)}$ means the missing one. Suppose $i \geq a(i)$. Then

$$\lambda_i(A+B) \leq \lambda_{a(i)}(A+B) \leq \lambda_{a(i)}(A) + \lambda_1(B) = \lambda_i(A+B)$$

and so $\lambda_{a(i)}(A+B) = \lambda_{a(i)}(A) + \lambda_1(B)$. This contradicts the assumption of case 2; so we must have $a(i) > i$. Because of the choice of a, we have $a(i+1) = i$. On the other hand, since $\lambda_{b(k)}(B) = 0$ for all $k \neq i$, b can be chosen such that $b(i+1) = 2$. Consequently we have $\lambda_{i+1}(A+B) = \lambda_i(A) + \lambda_2(B)$. Using Lemma 2.2(ii), there exists a common unit eigenvector x for A and B. Find a unitary matrix U with x as its first column. Then

$$U^*AU = \begin{bmatrix} \lambda_i(A) & 0 \\ 0 & C \end{bmatrix}, \qquad U^*BU = \begin{bmatrix} \lambda_2(B) & 0 \\ 0 & D \end{bmatrix},$$

where C and D are $(n-1) \times (n-1)$ Hermitian matrices, and $\operatorname{rank} D = 1$, since $\operatorname{rank} B = 1$. The spectra of C, D, and $C+D$ are respectively $\{\lambda_{a(k)}(A) : k \neq i+1\}$, $\{\lambda_{b(k)}(B) : k \neq i+1\}$, and $\{\lambda_k(A+B) : k \neq i+1\}$. Recall that $\lambda_k(A+B) = \lambda_{a(k)}(A) + \lambda_{b(k)}(B)$ for all k. Hence the eigenvalues of

$C+D$ are the sums of the eigenvalues of C and D in a suitable order, and so there exist permutations c and d of $\{1,\ldots,n-1\}$ such that $\lambda_k(C+D) = \lambda_{c(k)}(C)+\lambda_{d(k)}(D)$ for all $k=1,\ldots,n-1$. Therefore, $CD=DC$ by the induction assumption. Consequently, $AB=BA$. ∎

THEOREM 2.4. *If there exists a permutation b of $\{1,\ldots,n\}$ such that $\lambda_k(A+B)=\lambda_k(A)+\lambda_{b(k)}(B)$ for all $k=1,\ldots,n$, then $AB=BA$.*

Proof. We proceed by induction on n, the dimension of the matrices. For $n=1$, there is nothing to prove. Assume the statement is true when the dimension of the matrices is $n-1$. Suppose there exists a permutation b of $\{1,\ldots,n\}$ such that $\lambda_k(A+B)=\lambda_k(A)+\lambda_{b(k)}(B)$ for all $k=1,\ldots,n$. Choose k_0 such that $b(k_0)=1$, hence $\lambda_{k_0}(A+B)=\lambda_{k_0}(A)+\lambda_1(B)$. Using Lemma 2.2(i), there exists a common unit eigenvector x for A and B. Find a unitary matrix U with x as its first column. Then

$$U^*AU=\begin{bmatrix}\lambda_{k_0}(A) & 0\\ 0 & C\end{bmatrix},\qquad U^*BU=\begin{bmatrix}\lambda_1(B) & 0\\ 0 & D\end{bmatrix},$$

where C and D are $(n-1)\times(n-1)$ Hermitian matrices. The spectra of C, D, and $C+D$ are, respectively,

$$\begin{aligned}\lambda_1(A) &\geq \cdots \geq \widehat{\lambda}_{k_0}(A)\geq\cdots\geq\lambda_n(A),\\ \widehat{\lambda}_1(B) &\geq \lambda_2(B)\geq\cdots\geq\lambda_n(B),\\ \lambda_1(A+B) &\geq \cdots\geq\widehat{\lambda}_{k_0}(A+B)\geq\cdots\geq\lambda_n(A+B),\end{aligned}$$

where $\widehat{\lambda}_{k_0}(A),\widehat{\lambda}_1(B)$, and $\widehat{\lambda}_{k_0}(A+B)$ mean the missing ones. Define a permutation d of $\{1,\ldots,n\}$ as follows: for $1\leq k\leq k_0-1$, $d(k)=b(k)-1$; for $k_0\leq k\leq n-1$, $d(k)=b(k+1)-1$. Then it can be verified that $\lambda_k(C+D)=\lambda_k(C)+\lambda_{d(k)}(D)$ for all $k=1,\ldots,n-1$. Therefore, $CD=DC$ by the induction assumption. Consequently, $AB=BA$. ∎

A permutation π of $\{1,\ldots,n\}$ is called the identity permutation [respectively, the reverse permutation] if $\pi(k)=k$ [respectively, $\pi(k)=n+1-k$] for all $k=1,\ldots,n$.

COROLLARY 2.5. *Suppose there exist permutations a and b of $\{1,\ldots,n\}$ such that $\lambda_k(A+B)=\lambda_{a(k)}(A)+\lambda_{b(k)}(B)$ for all $k=1,\ldots,n$. If one of the permutations is either the identity permutation or the reverse permutation, then $AB=BA$.*

Proof. Because of Theorem 2.4, it suffices to consider the case when one of the permutation is the reverse permutation. Without loss of generality, we assume that $b(k) = n+1-k$ for all $k = 1, \ldots, n$. Moreover, a can be chosen such that, for $i < j$, $\lambda_{a(i)}(A) = \lambda_{a(j)}(A) \Rightarrow a(i) < a(j)$. Now for $1 \leq k \leq n-1$,

$$\begin{aligned}
\lambda_{a(k)} + \lambda_{n+1-k} &= \lambda_{a(k)} + \lambda_{b(k)} \\
&= \lambda_k(A+B) \\
&\geq \lambda_{k+1}(A+B) \\
&= \lambda_{a(k+1)} + \lambda_{b(k+1)} \\
&= \lambda_{a(k+1)} + \lambda_{n-k}
\end{aligned}$$

Since $\lambda_{n-k}(B) \geq \lambda_{n+1-k}(B)$, we have $\lambda_{a(k)}(A) \geq \lambda_{a(k+1)}(A)$. Consequently, $\lambda_{a(1)}(A) \geq \cdots \geq \lambda_{a(n)}(A)$, and hence $a(1) < \cdots < a(n)$, i.e., $a(k) = k$ for all $k = 1, \ldots, n$. Now apply Theorem 2.4 to have the conclusion that A and B commute. ∎

Since a permutation of $\{1, 2\}$ must be either the identity or the reverse permutation, we have the following corollary. It explains why there is no 2×2 Hermitian counterexample to the converse of Theorem 2.1.

COROLLARY 2.6. *Let A and B be 2×2 Hermitian matrices. If there exist permutations a and b of $\{1,2\}$ such that $\lambda_k(A+B) = \lambda_{a(k)}(A) + \lambda_{b(k)}(B)$ for $k = 1, 2$, then $AB = BA$.*

It is of interest to find all pairs of permutations a and b such that if $\lambda_k(A+B) = \lambda_{a(k)}(A) + \lambda_{b(k)}(B)$ for all $k = 1, \ldots, n$ then $AB = BA$. In this section we have given some examples and also a nonexample.

3. PROPERTY L

Two $n \times n$ (not necessary Hermitian) matrices A and B have property L if there exists a permutation π of $\{1, \ldots, n\}$ such that any real linear combination $tA + sB$ has as its eigenvalues $t\lambda_k(A) + s\lambda_{\pi(k)}(B)$. This definition was first suggested by M. Kac to study the commutativity of matrices. He also conjectured that two Hermitian matrices with property L commute. This was later confirmed by Motzkin and Taussky [4]. In this section we give an alternative proof using Theorem 2.4.

THEOREM 3.1. *If Hermitian matrices A and B have property L, then $AB = BA$.*

Proof. In order to simplify notation, let $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_n$ denote the eigenvalues of A and B, respectively. Since A and B have property L, there exists a permutation π such that, for any real t, the set of eigenvalues of $tA + B$ is $\{t\alpha_k + \beta_{\pi(k)} : k = 1, \ldots, n\}$. Without loss of generality, we can choose the permutation π such that

$$\alpha_k = \alpha_{k+1} \Rightarrow \beta_{\pi(k)} \geq \beta_{\pi(k+1)}.$$

If $\alpha_1 = \cdots = \alpha_n$, then A is a scalar, and hence $AB = BA$. Otherwise, take

$$t_0 = 1 + \max\left\{\left|\frac{\beta_{\pi(k+1)} - \beta_{\pi(k)}}{\alpha_k - \alpha_{k+1}}\right| : \alpha_k \neq \alpha_{k+1}\right\} > 0.$$

Then $t_0\alpha_1 + \beta_{\pi(1)} \geq \cdots \geq t_0\alpha_n + \beta_{\pi(n)}$. Hence $\lambda_k(t_0A + B) = \lambda_k(t_0A) + \lambda_{\pi(k)}(B)$. It follows from Theorem 2.4 that t_0A and B commute, and hence so do A and B since $t_0 \neq 0$. ■

4. PRODUCT OF TWO POSITIVE DEFINITE HERMITIAN MATRICES

Results analogous to those in Sections 1 and 2 are also true for the product of two positive definite Hermitian matrices. Throughout this section, A and B always denote $n \times n$ positive definite Hermitian matrices. The multiplicative counterparts of Weyl's inequalities are given below. These inequalities are not new. They can be seen as special cases of singular-value inequalities for the product of two matrices; for example see [2, p. 423].

THEOREM 4.1.

(i) *For integers i and j such that $1 \leq i, j, i+j-1 \leq n$,*

$$\lambda_{i+j-1}(AB) \leq \lambda_i(A)\lambda_j(B).$$

Moreover, equality holds if and only if there exists a unit vector x such that $Ax = \lambda_i(A)x$, $Bx = \lambda_j(B)x$, and $(AB)x = \lambda_{i+j-1}(AB)x$.

(ii) *For integers i and j such that $1 \leq i, j, i+j-n \leq n$,*

$$\lambda_{i+j-n}(AB) \geq \lambda_i(A)\lambda_j(B).$$

Moreover, equality holds if and only if there exists a unit vector x such that $Ax = \lambda_i(A)x$, $Bx = \lambda_j(B)x$, and $(AB)x = \lambda_{i+j-n}(AB)x$.

Proof. Since $A^{1/2}BA^{1/2}$ is positive definite Hermitian, it has a full set of orthonormal eigenvectors. Moreover, $\lambda_k(A^{1/2}BA^{1/2}) = \lambda_k(AB)$

for all $k = 1, \ldots, n$. Let $w_1, \ldots, w_n$ denote orthonormal eigenvectors of $A^{1/2}BA^{1/2}$ corresponding to the respective eigenvalues $\lambda_1(AB), \ldots, \lambda_n(AB); u_1, \ldots, u_n$ denote orthonormal eigenvectors of A corresponding to the respective eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$; and $v_1, \ldots, v_n$ denote orthonormal eigenvectors of B corresponding to the respective eigenvalues $\lambda_1(B), \ldots, \lambda_n(B)$.

Consider subspaces $S_1 = \text{span}\{u_i, \ldots, u_n\}$, $S_2 = A^{-1/2}[\text{span}\{v_j, \ldots, v_n\}]$, and $S_3 = \text{span}\{w_1, \ldots, w_{i+j-1}\}$. By Lemma 1.1, $\dim(S_1 \cap S_2 \cap S_3) \geq (n+1-i)+(n+1-j)+(i+j-1)-2n = 1$. Hence there exists a unit vector $y \in S_1 \cap S_2 \cap S_3$. Notice that $y = A^{-1/2}x$ for some $x \in \text{span}\{v_j, \ldots, v_n\}$. Using Lemma 1.2,

$$\begin{aligned} \lambda_{i+j-1}(AB) &\leq y^*(A^{1/2}BA^{1/2})y = x^*Bx \\ &\leq \lambda_j(B)x^*x = \lambda_j(B)y^*Ay \\ &\leq \lambda_j(B)\lambda_i(A). \end{aligned}$$

If $\lambda_{i+j-1}(AB) = \lambda_i(A)\lambda_j(B)$ then $y^*Ay = \lambda_i(A)$ and $x^*Bx = \lambda_j(B)x^*x$. By Lemma 1.2 again, $Ay = \lambda_i(A)y$ and $Bx = \lambda_j(B)x$. Since $y = A^{-1/2}x$, it follows that $x/\|x\|$ is a common unit eigenvector for A, B, and AB corresponding to the appropriate eigenvalues. The converse of the equality case is obvious. This completes the proof of (i). We omit the proof of (ii) because it is similar to (i) with some modifications like those in the proof of Theorem 1.3 (ii). ■

If X is a positive definite Hermitian matrix, then $\lambda_k(X^{-1}) = \lambda_{n+1-k}(X)^{-1}$ for all $k = 1, \ldots, n$. With this observation, (ii) of Theorem 4.1 also follows from applying (i) to matrices A^{-1} and B^{-1}. Using a continuity argument, one can easily deduce that both of the above inequalities still hold for positive semidefinite Hermitian matrices. From the proof, one also sees that the conclusion of the equality case requires only that at least one of the matrices be definite. The following example shows that this condition is essential. Consider the two positive semidefinite Hermitian matrices

$$A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $\lambda_2(AB) = \lambda_1(A)\lambda_2(B)$, but A and B do not commute, and hence they have no common eigenvector because of dimension 2. Of course, we have

THEOREM 4.2. *If $AB = BA$, then there exist permutations a and b of $\{1, \ldots, n\}$ such that $\lambda_k(AB) = \lambda_{a(k)}(A)\lambda_{b(k)}(B)$ for all $k = 1, \ldots, n$.*

Again the converse of it is not true in general. Here is an example. Let

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} a & 0 & f \\ 0 & b & e \\ f & e & c \end{bmatrix},$$

where $a = (21-3s)/8$, $b = (29-3s)/8$, $c = (7+3s)/4$, $f = \sqrt{(93s-347)/128}$, $e = \sqrt{(39s-129)/128}$, and $s = \sqrt{17}$. Then the spectra of A, B, and AB are respectively $\{3, 2, 1\}$, $\{5, 2, 1\}$, and $\{10, 3, 2\}$. Hence $10 = 2 \times 5$, $3 = 3 \times 1$, and $2 = 1 \times 2$, but $AB \neq BA$. However, we have the following partial converses. We omit their proofs because they are similar to the corresponding ones in Section 2.

THEOREM 4.3. *Suppose there exist permutations a and b of $\{1, \ldots, n\}$ such that $\lambda_k(AB) = \lambda_{a(k)}(A)\lambda_{b(k)}(B)$ for all $k = 1, \ldots, n$. If one of the permutations is the identity permutation or the reverse permutation, then $AB = BA$.*

COROLLARY 4.4. *Let A and B be 2×2 positive definite Hermitian matrices. If there exist permutations a and b of $\{1, 2\}$ such that $\lambda_k(AB) = \lambda_{a(k)}\lambda_{b(k)}(B)$ for $k = 1, 2$, then $AB = BA$.*

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REFERENCES

1 A. L. Duarte, *Desigualdades Espectrais e Problemas de Existencia em Teoria de Matrizes*, Ph.D. Thesis, Univ. of Coimbra, Coimbra, Portugal, 1992.

2 R. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge U.P., 1985.

3 Y. Ikebe, T. Inagaki and S. Miyamoto, The monotonicity theorem, Cauchy's interlace theorem, and the Courant-Fisher theorem, *Amer. Math. Monthly* 94:352–354 (1987).

4 T. S. Motzkin and O. Taussky, Pairs of matrices with property L, *Trans. Amer. Math. Soc.* 73:108–114 (1952).

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